A New Approach to Steepest-Ascent Trajectory Optimization

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The method of steepest ascent, or gradient method, has been applied with considerable success in recent years to the problem of determining the optimum trajectory for a prescribed vehicle. Use of the method usually requires the guessing of the desired increase in a performance parameter and/or certain weighting factors. Although it is usually easier for the engineer to estimate values for the forementioned parameters than for the Lagrangian multipliers required by the classical methods of the calculus of variations, the steepest-ascent method would be more usable if the requirements for guessing these were eliminated. The method described in this paper does eliminate the guessing of these parameters. A computer program using this method has been used to determine optimum trajectories for a wide variety of rocket-powered vehicles in several mission options. Convergence of the iteration is good, and the program can be used by engineers who are not familiar with the calculus of variations.

Nomenclature

D = drag force
g = acceleration of gravity
h = altitude
i = inclination of trajectory plane
L = lift force
R = planet radius
T = thrust
t = time
v = velocity
α = angle of attack and angle between thrust and velocity vectors
γ = flight path angle, referenced to local horizontal
λ = influence function
θ = angle between the local horizontal and the vehicle centerline
ω = rotation rate of planet

Introduction

THE primary difference between the present method and others described in the literature concerns the determination of the change in control (angle of attack) in the "optimizing" direction. The determination of the change in control required to correct errors in specified end conditions is the same as in other versions of the gradient method. In the case where the final velocity is to be maximized, for example, the "optimizing" angle-of-attack increment is usually determined by specifying a desired increase in velocity. In the method of this paper, we first determine approximate partial derivatives of final velocity with respect to the final values of the other variables. We then proceed to find the "optimum" angle of attack as if we were using the classical calculus of variations technique, i.e., from the partial derivatives we determine final values of the Lagrangian multipliers. These final values determine the values of the multipliers through the entire trajectory, and the "optimum" angle of attack can be solved for from the appropriate Euler or adjoint equation. The usual gradient techniques are used to determine the variations in final velocity which would occur as a result of specifying changes in the final values of the other variables, and these variations determine the partial derivatives mentioned previously.

Another factor that has been found useful in stabilizing the iteration in particularly sensitive cases is the use of attitude angle θ as the controlling parameter rather than angle of attack α. In order to use θ control, it is necessary to estimate the change in path angle γ which will occur at the next trajectory, because experience has shown that, when the θ control is determined by adding the new α to the current value of γ ($\theta = \gamma + \alpha$), the iteration will almost always *not* converge. Good convergence properties are obtained by determining the value of θ from $\theta = \gamma + \delta\gamma + \alpha_{new}$.

The techniques just described and others that are discussed in the paper result in a method for determining trajectories which is believed to combine the best features of the steepest ascent, or gradient method, and the classical methods of the calculus of variations.

A computer program using these new techniques has been developed for solving the rocket boost vehicle mission optimization problems that were analyzed in Ref. 1 (using the classical methods of the calculus of variations). As in Ref. 1, a variable coasting time may be included, and the optimum value is determined by the procedure. Effects of planet rotation, aerodynamic forces, variable atmosphere, time-variant thrust, and vehicle design or physiological limitations are included.

A numerical example is shown to illustrate the convergence properties of the current method in a somewhat difficult (sensitive) case.

Equations of Motion

Orientation of the rocket relative to the planet and the force, velocity, and acceleration vectors that exist during flight are shown in Fig. 1. Equations of motion used employ natural coordinates directed along and normal to the flight path:

$$m\dot{v} = T\cos\alpha - D - gm\sin\gamma \tag{1}$$

$$mv\{\dot{\gamma} - [v/(R+h)]\cos\gamma - 2\,\omega\cos i\} = T\sin\alpha + L - gm\cos\gamma \tag{2}$$

In addition, the kinematic relationship

$$\dot{h} = v\sin\gamma \tag{3}$$

is used. The Coriolis acceleration component ($2\omega\, v\cos i$) in the plane of motion is included in (2). Transverse Coriolis accelerations were excluded. Centrifugal terms of negligible magnitude have been deleted from the motion equations. Consideration of thrust-vector gimbaling was also excluded in order to eliminate an additional dependent variable. Hence, the variable α represents both the angle of attack and the angle between the thrust and velocity vectors.

In performing the analysis, it is assumed that the mass, thrust, lift, drag, and acceleration of gravity are known at all

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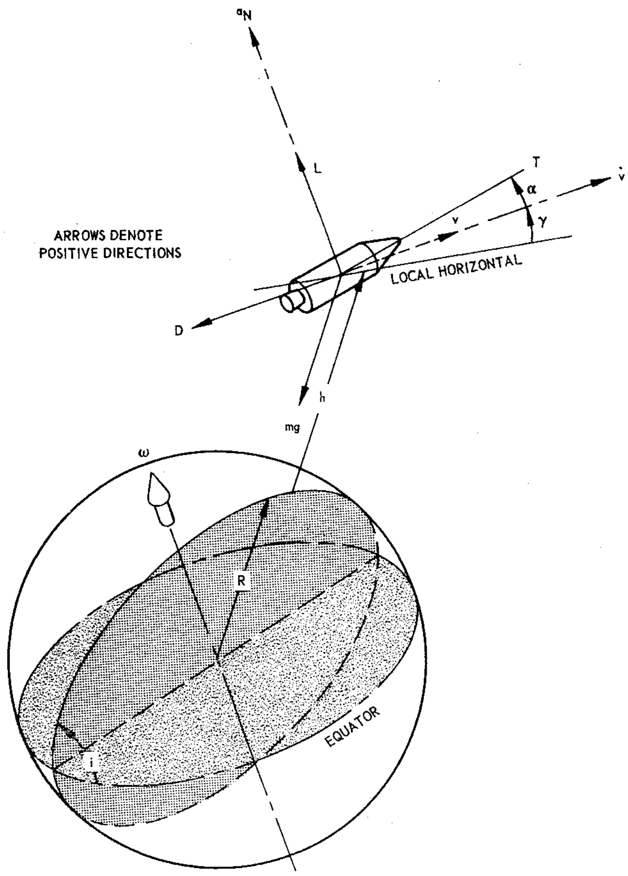

Fig. 1 Vehicle orientation.

points of the trajectory and have the following functional dependence: $m = m(t)$, $T = T(h, t)$, $L = L(v, h, \alpha)$, $D = D(v\, h, \alpha)$, $g = g(R + h)$. Lag effects in the action of the aerodynamic forces are excluded. The aerodynamic force coefficients may be discontinuous at staging points.

Basic Gradient Method

A more complete and quite clear derivation of the fundamental gradient or steepest-ascent method is given in Ref. 2. We will use similar notations, i.e., the differential equations describing the trajectory [Eqs. (1–3)] are represented by

$$dy_i/dt = f_i(y, \alpha, t) \qquad i = 1, 2, \ldots n \tag{4}$$

where, in our case, $n = 3$, $y_1 = v$, $y_2 = \gamma$, and $y_3 = h$.

We can calculate some nominal trajectory, which probably will not satisfy the desired boundary conditions, nor will it be optimal. Small perturbations around this nominal are described by

$$\frac{d}{dt}(\delta y_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} \delta y_j + \frac{\partial f_i}{\partial \alpha}\delta\alpha \tag{5}$$

The set of adjoint equations is, by definition,

$$\frac{d\lambda_i}{dt} = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial y_i} \lambda_j \tag{6}$$

Multiplying Eq. (5) by λ_i and Eq. (6) by δy_i, and summing, we get

$$\frac{d}{dt}\sum_{i=1}^{n} \lambda_i\, \delta y_i = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \alpha}\delta\alpha + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\lambda_i \frac{\partial f_i}{\partial y_j}\delta y_j - \lambda_j \frac{\partial f_j}{\partial y_i}\delta y_i\right) \tag{7}$$

Note that the double summation vanishes. This result justifies the choice of the adjoint equations (6). Now we are able to integrate Eq. (7) and obtain

$$\left[\sum_{i=1}^{n} \lambda_i\, \delta y_i\right]_{t_0}^{T} = \int_{t_0}^{T} \lambda_\alpha(t)\, \delta\alpha(t)\, dt \tag{8}$$

where

$$\lambda_\alpha = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \alpha}$$

Equation (8) is a fundamental relation in the gradient technique. If we wish to examine variations in any function $A[y(T)]$, we let

$$\lambda_i^A(T) = [\partial A/\partial y_i]_{t=T} \tag{9}$$

Obviously, then,

$$\left[\sum_{i=1}^{n} \lambda_i^A\, \delta y_i\right]_{t=T} = \delta A$$

and Eq. (8) becomes

$$\delta A = \int_{t_0}^{T} \lambda_\alpha^A(t)\, \delta\alpha(t)\, dt + \left[\sum_{i=1}^{n} \lambda_i^A\, \delta y_i\right]_{t=t_0} \tag{10}$$

Now we determine the effect of small changes in initial conditions or intermediate conditions (perturbations or errors) by replacing t_0 by t. These effects on the final values of the variables or the effects due to small changes in α can be determined from Eq. (10), with the boundary conditions (9) chosen so that the function A represents the variable or combination of variables in which we are interested.

The steepest-ascent (or descent) direction for an increase (or decrease) in the value of A is defined by

$$\delta\alpha(t) = K_A \lambda_\alpha^A(t) \tag{11}$$

where $K_A = \text{const.}$

It is desired to control the values of v_f, γ_f, and h_f to be obtained on the next iteration. In order to accomplish the desired changes with minimum over-all change in the control (α) program, we will use linear combinations of increments in the steepest-ascent directions for each of v_f, γ_f, and h_f, i.e.,

$$\delta\alpha = K_v\lambda_\alpha^v + K_\gamma\lambda_\alpha^\gamma + K_h\lambda_\alpha^h$$

Now if we define

$$I_{AB} = \int_{t_0}^{T} \lambda_\alpha^A(t)\ \lambda_\alpha^B(t)\ dt \tag{12}$$

we obtain from Eq. (10)

$$\delta v_f = K_v I_{vv} + K_\gamma I_{v\gamma} + K_h I_{vh} \tag{13}$$

$$\delta \gamma_f = K_v I_{v\gamma} + K_\gamma I_{\gamma\gamma} + K_h I_{\gamma h} \tag{14}$$

$$\delta h_f = K_v I_{vh} + K_\gamma I_{\gamma h} + K_h I_{hh} \tag{15}$$

assuming that the initial values are fixed. It should be noted that, on portions of the trajectory in which the angle of attack is actually being limited by an inequality constraint, the integral in Eq. (12) should be set to zero. This is because we are not free, because of the constraint, to choose $\delta\alpha$ according to Eq. (11). It is true that the portions of the next trajectory which will be constrained may not be the same as for the current trajectory unless we have almost converged on the desired solution. However, we do not as yet know in which direction α will be changed, and also it is important to keep Eqs. (13–15) linear.

In our basic problem, we wish to maximize $v_f [= v(T)]$ subject to the requirement of specified values for $\gamma_f [= \gamma(T)]$

and $h_f[= h(T)]$. The usual procedure is to specify $\delta\gamma_f$ and δh_f so that the specified values will be obtained and to estimate an increase δv_f in velocity. Then Eqs. (13–15) can be solved for K_v, $K\gamma$, and K_h, and we set

$$\delta\alpha = K_v\lambda_\alpha{}^v + K_\gamma\lambda_\alpha{}^\gamma + K_h\lambda_\alpha{}^h \tag{16}$$

and use the new α's to calculate a new nominal trajectory.

New Approach

The initial step in the current analysis involves considering separately the "optimizing" increments in α and the "boundary-condition" increments. There are two reasons for doing this. First, it is intended to lead to a technique that does not require the guessing of a value for δv_f. Second, as the optimum is approached, the denominator term obtained in the solution for Eqs. (13–15) approaches zero. Although this may mean that we are close enough to the optimum, it could cause real difficulty in some cases. This possibility is expected to occur in some realistic problems, because it has been found in several numerical examples that the denominator obtained in solving for the "boundary-condition" increments using only Eqs. (14) and (15) with $K_v = 0$ loses three or more significant figures of accuracy.

Part of the problem involved in estimating a value for δv_f arises from the change in end conditions. Considering the family of optimal trajectories with varying values of γ_f and h_f, there exist values of the partial derivatives of v_f with respect to γ_f and h_f. Thus, when the end conditions are changed, there is a related change in v_f, the quantity being optimized. If too small a value of δv_f is estimated, convergence to the optimum will be slow. If too large a value of δv_f is estimated, the optimum will be overstepped, although methods have been developed to prevent this.[3]

Since the "optimizing" increment is to be separate from the "boundary-condition" increment, it should cause little or no change in γ_f and h_f. The ratios $(K_\gamma/K_v)^\phi$ and $(K_h/K_v)^\phi$, which make $\delta\gamma_f = 0$ and $\delta h_f = 0$, can be determined from Eqs. (14) and (15). This then defines an "optimizing" function ϕ, where

$$\begin{aligned}\phi &= v + (K_\gamma/K_v)^\phi\gamma + (K_h/K_v)^\phi h \\ &= \lambda_{1_f}{}^\phi v + \lambda_{2_f}{}^\phi\gamma + \lambda_{3_f}{}^\phi h\end{aligned} \tag{17}$$

and

$$(\delta\alpha)_{\text{opt}} = K_\phi\lambda_\alpha{}^\phi \tag{18}$$

$$\lambda_\alpha{}^\phi = \lambda_\alpha{}^v + (K_\gamma/K_v)^\phi\lambda_\alpha{}^\gamma + (K_h/K_v)^\phi\lambda_\alpha{}^h \tag{19}$$

The values of the ratios are

$$\left(\frac{K_\gamma}{K_v}\right)^\phi = \frac{I_{vh}I_{\gamma h} - I_{v\gamma}I_{hh}}{I_{\gamma\gamma}I_{hh} - (I_{\gamma h})^2} \tag{20}$$

$$\left(\frac{K_h}{K_v}\right)^\phi = \frac{I_{v\gamma}I_{\gamma h} - I_{vh}I_{\gamma\gamma}}{I_{\gamma\gamma}I_{hh} - (I_{\gamma h})^2} \tag{21}$$

It is interesting to note that the same result for the function ϕ can be obtained in an entirely different manner. It is known from the classical method that

$$\lambda_{2_f}/\lambda_{1_f} = -\partial v_f/\partial\gamma_f \tag{22}$$

$$\lambda_{3_f}/\lambda_{1_f} = -\partial v_f/\partial h_f \tag{23}$$

Estimates of the changes in v_f due to changes in γ_f and h_f (caused by changing α in the gradient directions for γ_f and h_f) can be obtained by setting $K_v = 0$ in Eqs. (13–15). Then, by setting $\delta h_f = 0$ and eliminating $K\gamma$ and K_h, we can solve for $\delta v_f/\delta\gamma_f$ and similarly for $\delta v_f/\delta h_f$. Combining these results with Eqs. (22) and (23), we get the same results for $\lambda_{2_f}{}^\phi/\lambda_{1_f}{}^\phi$ and $\lambda_{3_f}{}^\phi/\lambda_{1_f}{}^\phi$ as were obtained using Eqs. (17, 20, and 21).

We still have not eliminated the requirement for estimating a performance improvement. It can now be done using a technique suggested, but not used, in Ref. 2. Instead of using Eq. (18), we regard

$$\lambda_\alpha{}^\phi = \sum_{i=1}^{n} \lambda_i{}^\phi \frac{\partial f_i}{\partial\alpha} = 0 \tag{8$'$}$$

as a transcendental equation for α and solve for the "optimizing" α at each point along the flight path. This could not be done before determining the proper function ϕ, because the solutions of the transcendental equation for the "optimum" α cannot be added, and obviously the converged solution must agree with the classical results, which require that Eqs. (22) and (23) be satisfied by the set of λ's that are used to determine the optimum α's.

Because the influence functions $\lambda_i{}^\phi$'s are likely to change slightly on the next trajectory, the best convergence has been found to occur when the angle of attack used is the average of the previous value and the value determined from Eq. (8′), plus the increments required to correct the end conditions. Thus,

$$(\delta\alpha)_{\text{opt}} = \tfrac{1}{2}(\alpha_{\text{opt}} - \alpha_{\text{old}}) \tag{24}$$

This factor of $\frac{1}{2}$ was also found to provide improved convergence by Breakwell in Ref. 5, although this result was not included in the paper but was discussed in the verbal presentation. Convergence to the optimum is considered adequate when the maximum value of the difference between α_{opt} and α_{old} is less than some value. A tolerance value of 0.5° appears to be adequate from evaluation of numerical results.

The "optimizing" α increment obtained by this method is equivalent to replacing the constant K_ϕ in Eq. (18) by a variable $K_\phi(t)$. The variable $K_\phi(t)$ is always positive when α_{opt} and α_{old} differ by less than π for the assumed force representation. Thus, Eq. (24) automatically determines the weighting matrix that other gradient methods require to speed convergence. This method, however, does not apply the weighting matrix to the "boundary-condition" increments where it is neither required nor desirable.

Use of Eq. (24) in place of Eq. (18) means that the "optimizing" increment will cause changes in γ_f and h_f which must be taken into account, although the changes usually are small. The changes can be evaluated with a negligible increase in computer time using Eq. (10). Use of Eq. (24) actually improves the convergence toward the optimum relative to using Eq. (18) with the best possible constant value of K_ϕ. Equation (24) provides a more realistic variation of $(\delta\alpha)_{\text{opt}}$ with time. Consider, for example, a multistage rocket vehicle. The $\lambda_i{}^\phi$'s are continuous across a staging point. Neglecting aerodynamic forces, the optimum α is also continuous across the staging point (this is approximately true when aerodynamic forces are considered). But the value of $\lambda_\alpha{}^\phi$ is almost directly proportional to the thrust-to-weight ratio, and use of Eq. (18) tends to cause large discontinuities in the angle of attack when there should be none. Continuous, but inaccurate, variations within a given stage are obtained for the same reason if Eq. (18) is used.

The "boundary-condition" increments are determined by solving Eqs. (14) and (15) for K_γ and K_h with $K_v = 0$. Then

$$(\delta\alpha)_{bc} = K_\gamma\lambda_\alpha{}^\gamma + K_h\lambda_\alpha{}^h$$

Free Boundary Values

We will consider the case where the value of γ_f is specified, but h_f is free. This is applicable to such missions as maximum ballistic range or maximum apogee altitude[1] where the trajectory is integrated only to burnout. In place of the bound-

Table 1 Boost rocket characteristics

Thrust and weight data	Stage 1	Stage 2
Rated thrust, lb $\times 10^{-3}$ (vacuum)	1200	500
Burning time, sec	210	306
Nozzle exit area, ft^2	0	0
Specific impulse, sec (vacuum)	420	900
Stage weight, lb $\times 10^{-3}$	650	350
Propellant weight, lb $\times 10^{-3}$	600	170

Mach	Aerodynamic data[a] C_{D0}	$C_{L\alpha}$
0	0.425	11.6
0.5	0.425	11.8
0.9	0.425	13.3
1.2	1.185	14.5
3.0	0.791	9.4
6.0	0.609	8.1
10.0	0.560	8.0
15.0	0.560	7.3

[a] S = 70 ft^2, ARDC atmosphere.

ary condition on h_f, we have a boundary condition on $\partial v_f/\partial h_f$ or $\lambda_{3f}{}^{\phi}/\lambda_{1f}{}^{\phi}$. The value of $\lambda_{1f}{}^{\phi}$ is always taken to be unity. The value of $\lambda_{3f}{}^{\phi}$ is determined from the boundary condition, and $\lambda_{2f}{}^{\phi}$ is chosen so that $I_{\phi\gamma} = 0$. This is equivalent to writing Eq. (14) in the form

$$\delta\gamma_f = 0 = I_{v\gamma} + \lambda_{2f}{}^{\phi} I_{\gamma\gamma} + \lambda_{3f}{}^{\phi} I_{\gamma h} \tag{25}$$

and solving for $\lambda_{2f}{}^{\phi}$. The last term does not drop out unless the final altitude is completely free, which then makes $\lambda_{3f}{}^{\phi} = 0$. Setting $I_{\phi\gamma} = 0$ would make the functions ϕ and γ orthogonal if Eq. (18) were used to determine the "optimizing" increment instead of Eq. (24). Since Eq. (24) is used to determine $(\delta\alpha)_{\text{opt}}$, the correction required to orthogonalize, or prevent a change in γ_f, must be calculated. This method minimizes the orthogonalizing correction, which is important for good convergence toward the optimum. A method that was tried and discarded was to determine $\lambda_{2f}{}^{\phi}$ from Eq. (22) as is done when both γ_f and h_f are specified. This required much larger orthogonalizing corrections, which sometimes opposed the optimization process.

It has been found numerically that use of Eq. (25) gives a value of $\lambda_{2f}{}^{\phi}$ which is very nearly correct for the conditions 1) γ_f equal to the current value and 2) h_f changed such that the desired value for $\lambda_{3f}{}^{\phi}$ is attained; i.e., it appears that Eqs. (14) and (15) give good estimates of the rate of change of the $\lambda_{if}{}^{\phi}$'s with respect to each other for $\delta\gamma_f = 0$ [Eq. (14)] or $\delta h_f = 0$ [Eq. (15)], where the K_i's are replaced by the respective $\lambda_{1f}{}^{\phi}$'s.

In this case, the "boundary-condition" increment is simply

$$(\delta\alpha)_{bc} = K_\gamma \lambda_\alpha{}^\gamma \tag{26}$$

where

$$K_\gamma = \delta\gamma_f / I_{\gamma\gamma} \tag{27}$$

and $\delta\gamma_f$ is the desired change in γ_f minus the change caused by the "optimizing" increment.

Attitude Angle Control

The entire analysis of this paper is based upon the assumption of linear variations about a nominal trajectory. In some cases the variations in control (α) may result in significantly nonlinear variations in the state variables. Characteristics that tend to produce nonlinear variations are long burning times, significant lift forces (which tend to cause skipping), low initial velocities, and γ values near 90°. In the few cases where nonlinear effects caused a noticeable effect, it appeared that the effect arose primarily through the path angle γ. Thus, if we could approximately linearize the variations in γ, the nonlinearity problem would be significantly reduced.

The use of attitude angle control will cause the achieved values of γ vs time to follow rather closely an expected curve of γ vs time. Since $\theta = \gamma + \alpha$ and θ is defined as a function of time, if γ increases relative to the expected value, α will decrease, causing a decrease in $d\gamma/dt$. The decrease in $d\gamma/dt$ then tends to turn the γ value back toward the expected value.

At first, the use of θ control was tried by setting

$$\alpha_{\text{new}} = \alpha_{\text{old}} + (\delta\alpha)_{\text{opt}} + (\delta\alpha)_{bc} \tag{28}$$

and

$$\theta_{\text{new}} = \gamma + \alpha_{\text{new}} \tag{29}$$

In Eq. (29), the new α's calculated for the next trajectory are added to the current values of γ to determine the new θ values. It was found that the iteration practically never converged using this scheme, even for problems that converged readily using α control. This was because the expected linear variations in γ are not considered in Eq. (29).

A method for calculating the expected change in γ vs time was included in the computer program, Eq. (29) was replaced by

$$\theta_{\text{new}} = \gamma + \delta\gamma + \alpha_{\text{new}} \tag{30}$$

and it was found that this method was always as good as or better than using α control. Now the use of θ control stabilized the trajectories and provided convergence in sensitive cases that would not converge otherwise.

The values of $\delta\gamma$ in Eq. (30) were evaluated by considering each point in the trajectory as a final point. Equation (10) becomes

$$\delta\gamma(t) = \int_{t_0}^{t} \lambda_\alpha{}^A (\alpha_{\text{new}} - \alpha) dt \tag{31}$$

where

$$\lambda_1{}^A(t) = 0 \qquad \lambda_2{}^A(t) = 1 \qquad \lambda_3{}^A(t) = 0 \tag{32}$$

It is not necessary to evaluate the many new sets of influence functions vs time arising from the application of Eq. (32) at each point in the trajectory. Instead, we will make use of linear combinations of the three sets of influence functions which have already been calculated. Starting from the initial point of the trajectory, we evaluate

$$\begin{aligned} \delta F_1(t) &= \int_{t_0}^{t} \lambda_\alpha{}^v (\alpha_{\text{new}} - \alpha) dt \\ \delta F_2(t) &= \int_{t_0}^{t} \lambda_\alpha{}^\gamma (\alpha_{\text{new}} - \alpha) dt \\ \delta F_3(t) &= \int_{t_0}^{t} \lambda_\alpha{}^h (\alpha_{\text{new}} - \alpha) dt \end{aligned} \tag{33}$$

and at each point we get $\delta\gamma(t)$ from

$$\delta\gamma(t) = C_1(t)\,\delta F_1(t) + C_2(t)\,\delta F_2(t) + C_3(t)\,\delta F_3(t) \tag{34}$$

where the coefficients C_i are determined at each point from

$$\begin{aligned} \lambda_1(t) &= 0 = C_1(t) \quad \lambda_1{}^v(t) + C_2(t) \quad \lambda_1{}^\gamma(t) + C_3(t)\,\lambda_1{}^h(t) \\ \lambda_2(t) &= 1 = C_1(t) \quad \lambda_2{}^v(t) + C_2(t) \quad \lambda_2{}^\gamma(t) + C_3(t)\,\lambda_2{}^h(t) \\ \lambda_3(t) &= 0 = C_1(t) \quad \lambda_3{}^v(t) + C_2(t) \quad \lambda_3{}^\gamma(t) + C_3(t)\,\lambda_3{}^h(t) \end{aligned} \tag{35}$$

The use of Eqs. (33–35) is equivalent to using Eqs. (31) and (32) and requires only a small fraction of the number of computations which would result if Eqs. (31) and (32) were used directly.

Variable Coasting Time

Considering the problem of variable coasting time, it can be shown[1] that the additional boundary condition (arising from the transversality condition) is of the form $\mu_c\,\delta t = 0$. This requires that the function μ_c be equal to zero. The function μ_c is of the form

$$\mu_c = \sum_{i=1}^{n} \lambda_i{}^{\phi}\,\dot{y}_i$$

where μ_c is evaluated during the coasting period. It can also be shown that the value of μ_c does not change during the coasting period, since, when the vehicle is not thrusting, none of the variables or parameters associated with the motion are prespecified as functions of time. Thus, a change in the coasting time has absolutely no effect on the boundary condition associated with that coasting time, and the controls during the powered portions of the trajectory must be manipulated in order to satisfy the boundary condition arising from the arbitrary (i.e., optimum) coasting time.

In the current procedure, the control variable during the powered stages is varied so as to satisfy the conditions of final altitude specified and $\mu_c = 0$, whereas for each trajectory the coasting period is terminated when the path angle is equal to the desired final value minus the change in path angle during last-stage boost, which was calculated on the previous trajectory. The boost trajectory preceding the variable coasting period is treated by the steepest-ascent method, with the desired value of γ at the start of coast determined so that the (two-body) change in apogee altitude equals the desired change in final altitude. This approximation is sufficiently accurate for most cases, even when $\gamma_f \neq 0$ and the aerodynamic forces during coasting are significant. The value of h at the start of coast is free; the boundary condition is that $\mu_c = 0$. The variable coasting stage and succeeding stages are calculated using the classical method of the calculus of variations. The initial multipliers used are the values calculated for the end point of the trajectory prior to coast, but with both λ_2 and λ_3 adjusted according to Eq. (25) as required to make $\mu_c = 0$ (i.e., $\delta\lambda_{2_f} I_{\gamma\gamma} + \delta\lambda_{3_f} I_{\gamma h} = 0$). The adjustment to set $\mu_c = 0$ makes the values of α small during and after the variable coasting period and prevents large changes from one trajectory to the next. It has been found that this method properly relates cause and effect and provides rapid convergence, usually within 10 trajectories.

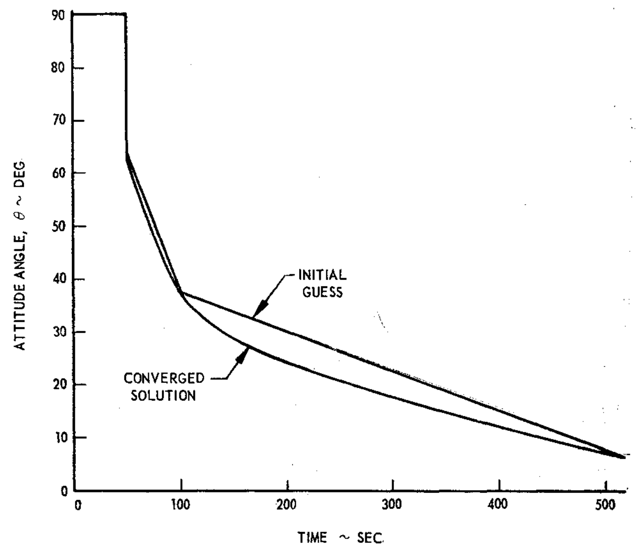

Fig. 3 Attitude angle vs time.

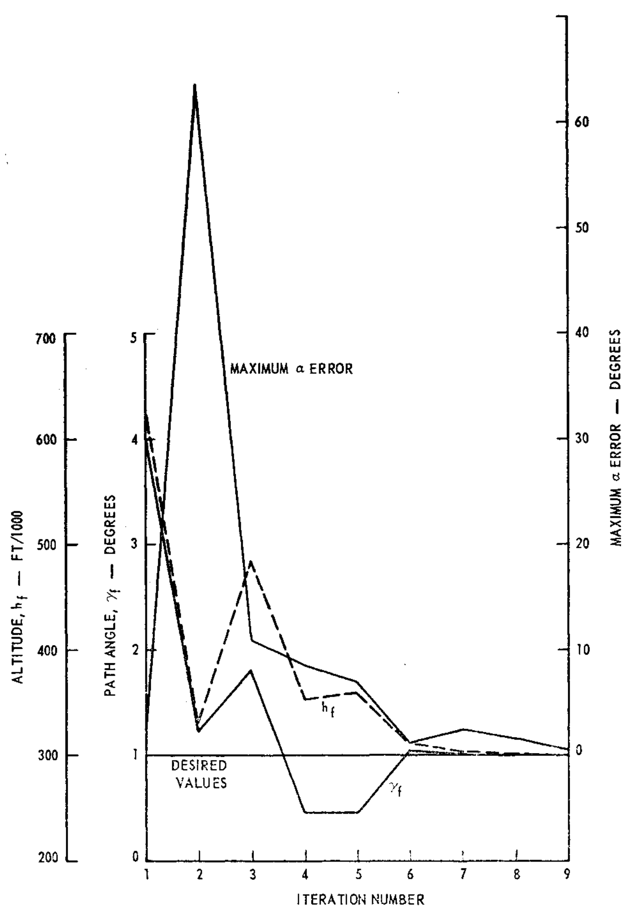

Fig. 2 Convergence of a sample problem.

Numerical Example

Characteristics of the rocket vehicle used in the example are shown in Table 1. The rocket and the boundary conditions were selected to illustrate the convergence of the method under adverse conditions: extremely long burning time, significant lift forces, low initial velocity, vertical initial path angle, and burnout at near-circular speed, relatively low altitude, and positive path angle.

Because of the sensitivity of the trajectory in this example, the iteration did not converge on the first attempt. The boundary-condition errors were being reduced efficiently, but the maximum α error was increasing. This indicated that the standard factor of $\frac{1}{2}$ in Eq. (24) was too large for this particular case. The good convergence illustrated in Fig. 2 resulted when the value of $\frac{1}{2}$ was replaced in Eq. (24) by 0.2. This is not typical; i.e., in all but the most sensitive cases, the standard value of $\frac{1}{2}$ works best in Eq. (24).

At the end of the arbitrarily specified 50-sec vertical initial segment, the velocity was 470 fps and the altitude 10,470 ft. The starting guess for the control θ is shown in Fig. 3, along with the converged solution. The initial guess consisted merely of three sets of θ, time values with linear interpolation used at intermediate points. The final velocity achieved on the converged trajectory was 26,401 fps, or 83.45% of the ideal velocity. The iteration shown in Fig. 2 required less than 2 min of IBM 7090 running time.

Conclusions

A practical computer procedure has been developed for determining the optimum angle-of-attack program for a variety of rocket boost mission options. The procedure can be easily used by engineers who are not familiar with trajectory optimization techniques as it does not require the guessing of any abstract parameters such as Lagrangian multipliers or even an estimated performance increase. The method has more reliable convergence properties than the classical method of Ref. 1 but uses certain features of the classical method to provide easier and faster convergence than the usual gradient method.

The techniques described herein could readily be combined with the techniques recently developed by Denham and Bryson[3] for handling inequality constraints on the state variables. The techniques of Denham and Bryson appear to provide a significant improvement in ease of use and speed of convergence relative to other known methods of accounting for state variable inequality constraints. Combination of the two methods would result in a procedure that does not require the guessing of either a step size in the gradient direction or a weighting factor for the constraint penalty function.

It is not known at present whether the current method would be advantageous for the difficult airplane-type performance optimizations for which Kelley was able to obtain solutions,[4] although the method has been successfully applied to the problem of maximizing the range of a gliding vehicle with both the initial and final conditions differing appreciably from equilibrium glide conditions.

Application of this method for determining the "optimizing" increment to generalized gradient problems (i.e., other than trajectory optimization) depends largely upon the form that Eq. (8′) takes. The problem must be related to a classical calculus of variations problem, and the appropriate scheme for determining the optimum control from known values of the Lagrangian multipliers must be available.

For rocket vehicles, at least, the current method appears to incorporate the best features of both the steepest-ascent and classical calculus of variations methods. In the present method, the restriction imposed in Ref. 1, that the burnout path angle be specified in the maximum apogee altitude mission, is removed because of the improved convergence properties. The numerical example illustrates convergence in a case where it would be difficult, if not impossible, to obtain the desired solution using the method of Ref. 1. The use of attitude angle control in the manner described herein is an important factor in the improved convergence properties

The attitude angle control developed herein may be viewed as a "zero-order" feedback control or a greatly simplified approach having the same purpose as the closed-loop guidance technique described in Ref. 6.

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